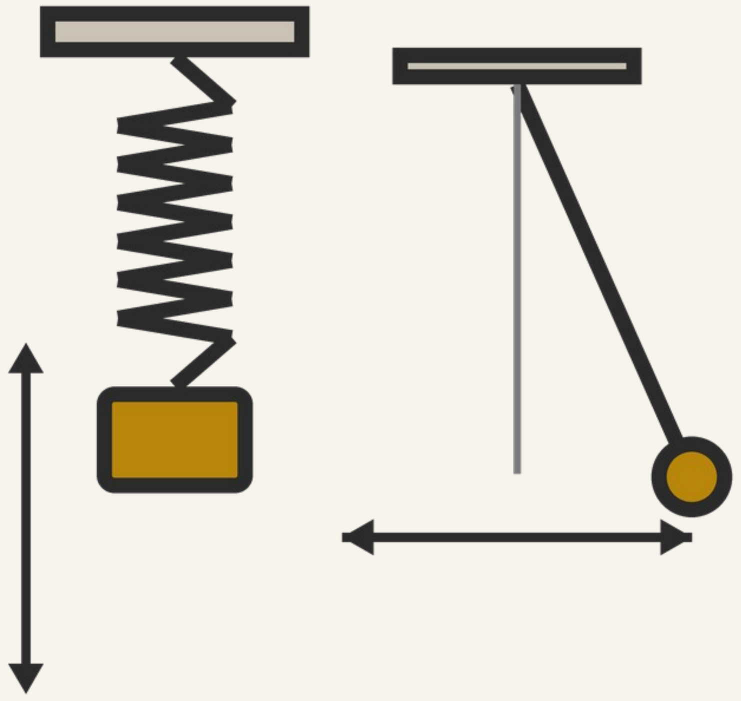
QUANTA AND WAVES · 2.3 SHM

Simple Harmonic Motion

Pull it, let it go — and the maths tells you where it is at every instant.

Advanced Higher Physics



LEARNING INTENTION

To define simple harmonic motion, derive its equations of motion and energy, and use them to solve problems.

By the end of this topic…

1. I can state the definition of SHM from memory.
2. I can derive $a = -\omega^2 y$ from $y = A \sin \omega t$ and from $y = A \cos \omega t$.
3. I can derive $v = \pm\omega\sqrt{(A^2 - y^2)}$ and $E_k = \frac{1}{2}m\omega^2(A^2 - y^2)$.
4. I can derive $T = 2\pi\sqrt{(L/g)}$, and find T for any SHM system.
5. I can describe underdamping, critical damping and overdamping.

PERIOD 1 · DEFINING SHM

When is motion simple harmonic?

One condition decides it.

Systems that oscillate with SHM

Each stores potential energy and has mass to move

System	Stores E_p as	E_k of
Mass on a spring	elastic energy of the spring	the mass
Simple pendulum	gravitational PE of the bob	the bob
Trolley and two springs	elastic energy of the springs	the trolley
Tube floating in water	gravitational PE of the tube	the tube

Definition of simple harmonic motion

JOTTER NOTES

The unbalanced force, and therefore the acceleration, are proportional to the displacement from the rest position and act in the opposite direction to it. $F = -ky$

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JOTTER NOTES

The restoring force

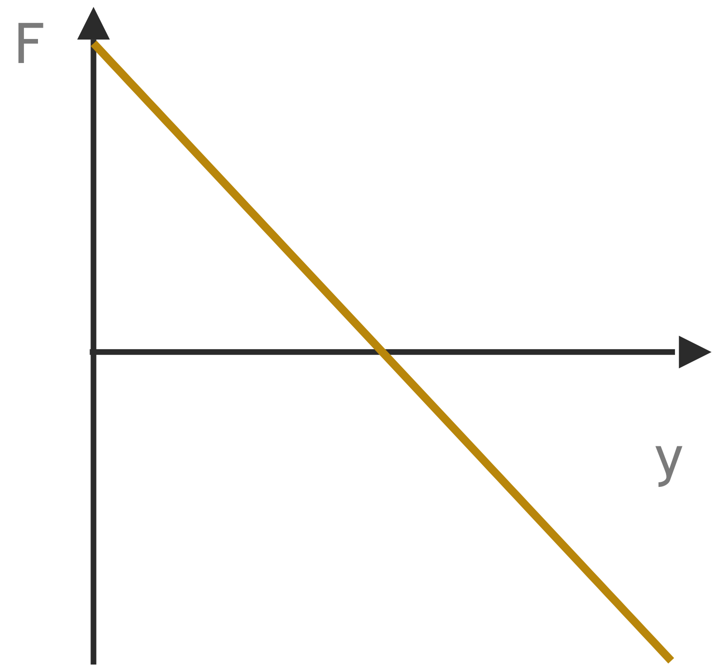
For a spring, the restoring force obeys Hooke's law:

$$F = -ky$$

k is the force constant, in N m^{-1}.

The minus sign means the force always points back

towards the rest position.



Angular frequency

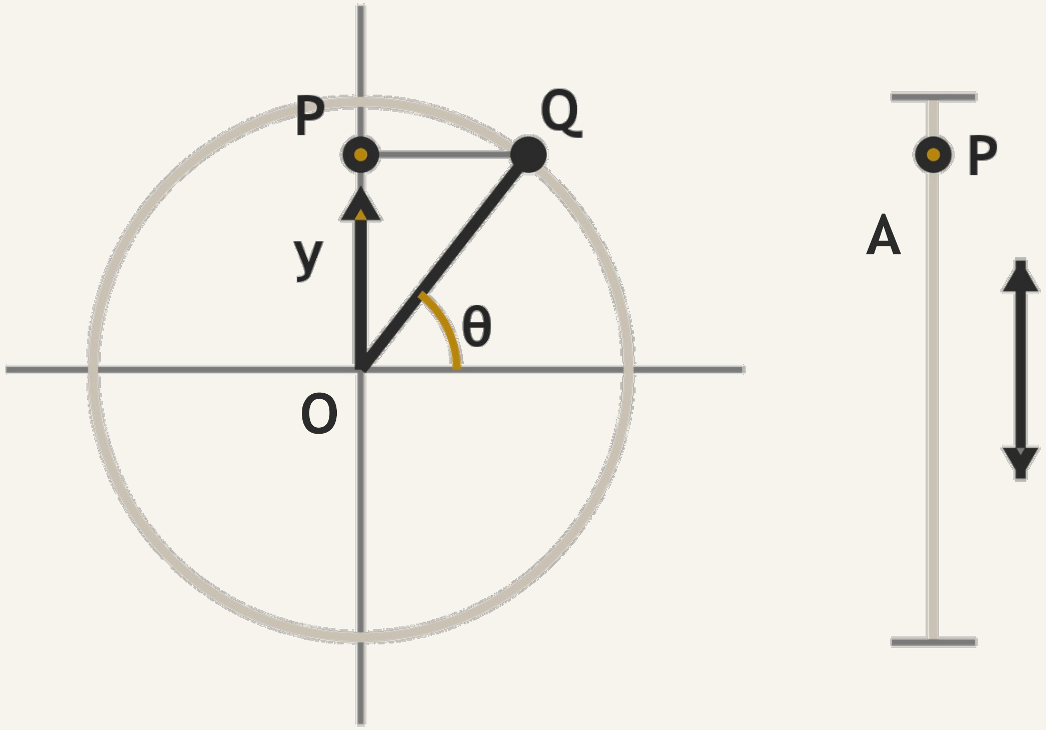
JOTTER NOTES

$$\omega = 2\pi f \qquad \omega = \frac{2\pi}{T}$$

ω	f	T
angular frequency, rad s^{-1}	frequency, Hz	period, s

ω is the angular frequency of the oscillation.

SHM and circular motion



Q circles at constant ω; its shadow P performs SHM, and $\theta = \omega t$.

From Hooke's law to the SHM condition

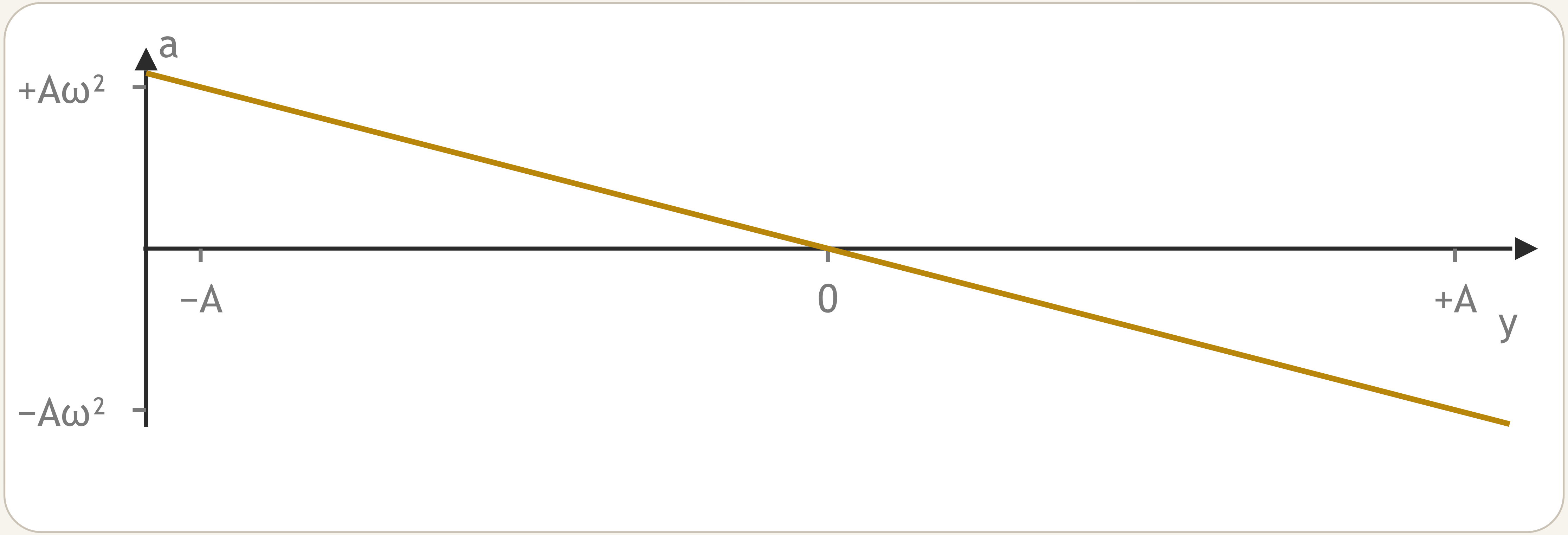
$F = -ky$	The restoring force on the mass.
$F = ma$	Newton's second law for that force.
$ma = -ky$	One force, so equate the two.
$a = -\frac{k}{m}y$	Divide both sides by the mass m.
$a = -\omega^2 y$	k/m is positive — it is ω^2.

So for a mass on a spring, $\omega^2 = k/m$.

Acceleration against displacement

JOTTER NOTES

A straight line through the origin, gradient $-\omega^2$



Worked example 1 — 2018 Q9(a)

A ball-bearing on a track makes 1·5 oscillations in 2·5 s.
Its amplitude is 0·20 m. Find ω and the maximum speed.

oscillations = 1·5

$t = 2{\cdot}5\ \text{s}$

$A = 0{\cdot}20\ \text{m}$

$\omega = ?$

$v_{max} = ?$

$$T = \frac{2\cdot 5}{1\cdot 5} = 1\cdot 67\ s$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{1\cdot 67}$$

$\omega = 3{\cdot}8\ \text{rad s}^{-1}$

$$v = \pm\omega\sqrt{A^2 - y^2}$$

$\underline{v = \pm 3{\cdot}8 \times 0{\cdot}20 = \pm 0{\cdot}76\ \text{m s}^{-1}}$

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Starter — from memory

Close your notes. Write full sentences, not bullet points.

1. State the definition of simple harmonic motion.
2. Define the moment of inertia of an object.
3. State the principle of conservation of angular momentum.
4. Write ω in terms of f, and in terms of T.
5. Derive $v = u + at$ using calculus.

PERIOD 2 · EQUATIONS OF MOTION

Where the equations come from

What function has a second derivative of $-\omega^2$ times itself?

$y = A \sin \omega t$ gives $a = -\omega^2 y$

JOTTER NOTES

MANDATORY DERIVATION — you must reproduce every line

$y = A \sin \omega t$	Start from the displacement,
$v = \dfrac{dy}{dt}$	Velocity is the rate of change of y.
$v = \omega A \cos \omega t$	sin ωt differentiates to ω cos ωt.
$a = \dfrac{dv}{dt} = \dfrac{d^2y}{dt^2}$	Acceleration: rate of change of v.
$a = -\omega^2 A \sin \omega t$	cos ωt differentiates to −ω sin ωt.
$a = -\omega^2 y$	A sin ωt is y — substitute.

y = A cos ωt gives a = −ω²y

JOTTER NOTES

MANDATORY DERIVATION — you must reproduce every line

$y = A\cos\omega t$	Start again, with y = A at t = 0.
$v = \dfrac{dy}{dt}$	The same first step: differentiate.
$v = -\omega A\sin\omega t$	cos ωt differentiates to −ω sin ωt.
$a = \dfrac{dv}{dt} = \dfrac{d^2y}{dt^2}$	Acceleration: rate of change of v.
$a = -\omega^2 A\cos\omega t$	−sin ωt differentiates to −ω cos ωt.
$a = -\omega^2 y$	A cos ωt is y — substitute.

Deriving v = ±ω√(A² − y²) — part 1

JOTTER NOTES

MANDATORY DERIVATION — you must reproduce every line

$y = A \sin \omega t$	The displacement from the equation sheet.
$v = \omega A \cos \omega t$	Its velocity, from the same derivation.
$\sin \omega t = \dfrac{y}{A}$	Rearrange the first equation for sin ωt.
$\cos \omega t = \dfrac{v}{\omega A}$	Rearrange the second equation for cos ωt.

Both are now in terms of y and v. The identity comes next.

Deriving $v = \pm\omega\sqrt{(A^2 - y^2)}$ — part 2

JOTTER NOTES

MANDATORY DERIVATION — you must reproduce every line

$sin^2\omega t + cos^2\omega t = 1$	The identity that links them.
$\frac{y^2}{A^2} + \frac{v^2}{\omega^2 A^2} = 1$	Substitute both into the identity.
$\omega^2 y^2 + v^2 = \omega^2 A^2$	Multiply every term by $\omega^2 A^2$.
$v^2 = \omega^2(A^2 - y^2)$	Subtract $\omega^2 y^2$; factorise out ω^2.
$v = \pm\omega\sqrt{A^2 - y^2}$	Square root both sides. The $\pm$ is there because the mass passes each point in both directions.

JOTTER NOTES

Velocity at the ends and at the middle

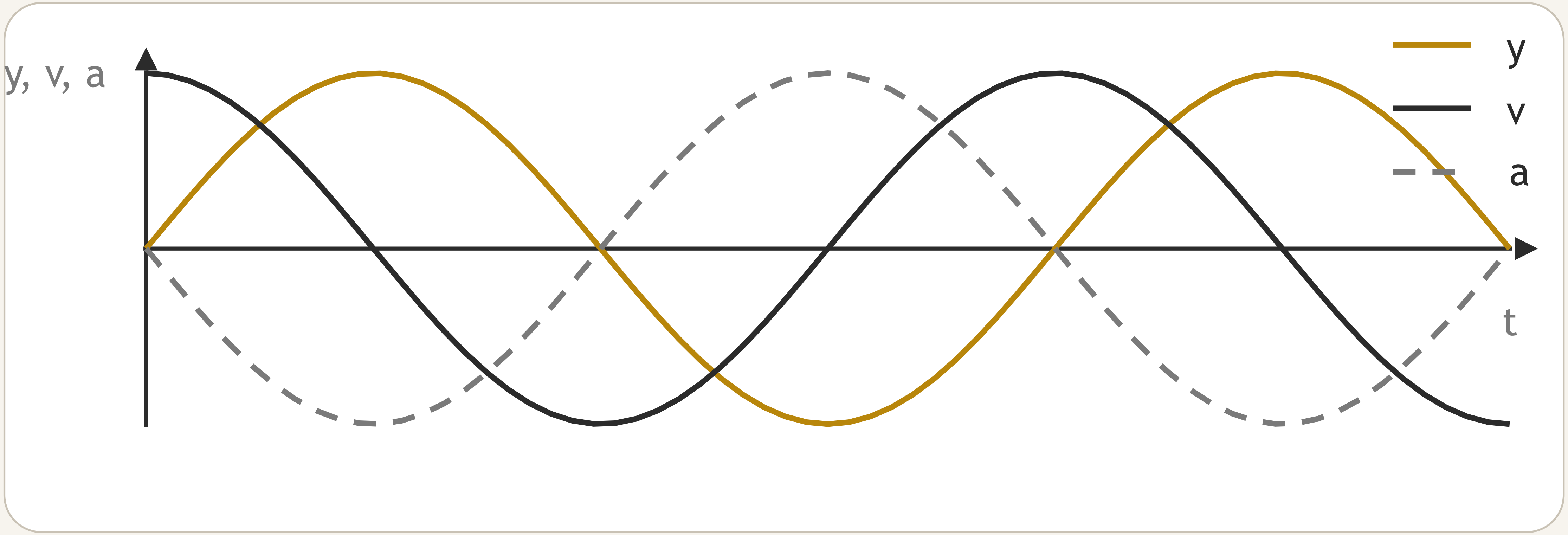
$y = \pm A$ into the relationship: the square root is zero, so the mass is momentarily at rest at each extreme.

$y = 0$ into it: the speed is at its maximum, $v = \pm \omega A$, as the mass passes through the rest position.

Displacement, velocity and acceleration

JOTTER NOTES

v leads y by a quarter period; a is out of phase with y



Starter

1. State the definition of simple harmonic motion.
2. Derive $a = -\omega^2 y$ from $y = A \sin \omega t$.
3. Derive $v = \pm\omega\sqrt{(A^2 - y^2)}$.
4. State where v is greatest, and where a is greatest.
5. Define the moment of inertia of an object.

PERIOD 3 · ENERGY

Where the energy goes

Total energy is constant, whatever the displacement.

JOTTER NOTES

The energy of an oscillator

As the mass oscillates, energy moves between kinetic and potential. At any displacement y,

$$E_k = \frac{1}{2}m\omega^2(A^2 - y^2)$$

and the potential energy at that displacement is

$$E_p = \frac{1}{2}m\omega^2 y^2$$

Deriving $E_k = \frac{1}{2}m\omega^2(A^2 - y^2)$

JOTTER NOTES

MANDATORY DERIVATION — you must reproduce every line

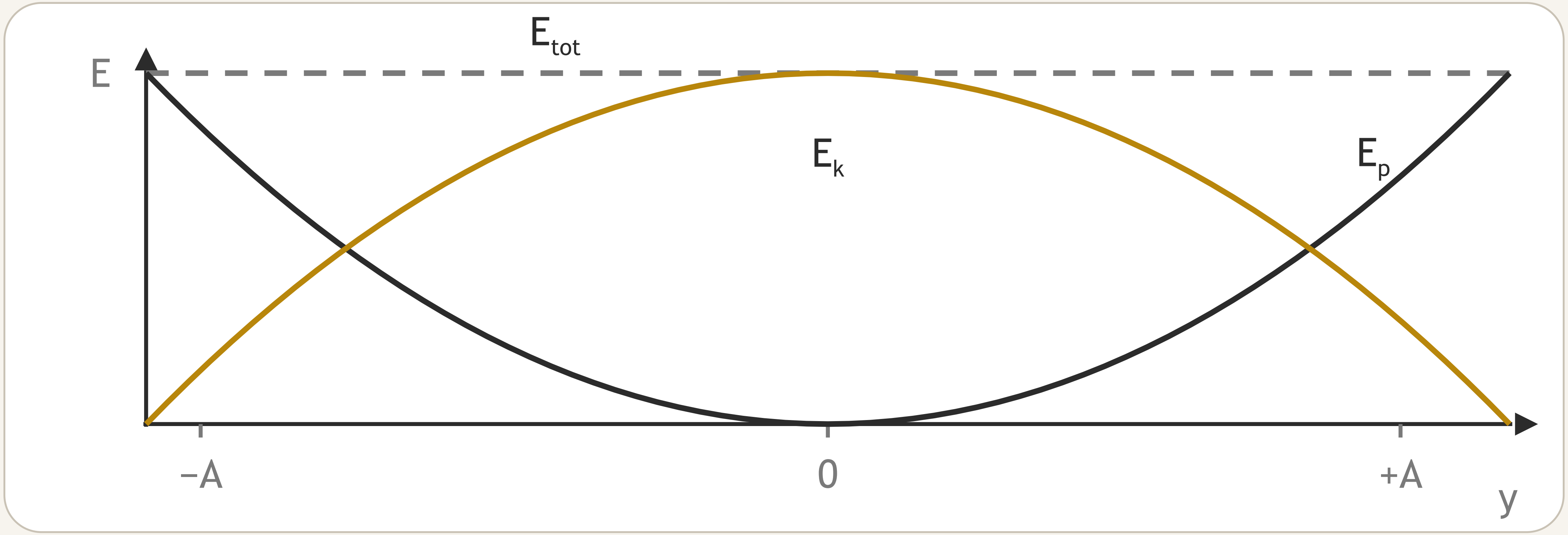
$E_k = \frac{1}{2}mv^2$	Kinetic energy of the mass.
$v = \pm\omega\sqrt{A^2 - y^2}$	Velocity at any displacement y.
$v^2 = \omega^2(A^2 - y^2)$	Square it; the ± disappears.
$E_k = \frac{1}{2}m \times \omega^2(A^2 - y^2)$	Substitute for v^2.
$E_k = \frac{1}{2}m\omega^2(A^2 - y^2)$	Tidy up — the required relationship.

E_k is greatest at $y = 0$, and zero at $y = \pm A$.

Energy against displacement

JOTTER NOTES

The two parabolas always add to the same total



JOTTER NOTES

Potential energy and total energy

Take the potential energy to be zero at the rest position.

The energy the mass loses as kinetic it gains as potential:

$$E_p = \frac{1}{2}m\omega^2y^2$$

Adding E_k and E_p gives a total that does not depend on y:

$$E_{tot} = \frac{1}{2}m\omega^2A^2$$

Worked example 2 — 2019 Q11(a)

A tin of paint, mass 3·67 kg, is shaken 580 times per minute. Its amplitude is 0·013 m. Find ω and the maximum E_k.

$m = 3{\cdot}67$ kg

$f = 580$ per minute

$A = 0{\cdot}013$ m

$\omega = ?$

$E_k = ?$

$$f = \frac{580}{60} = 9\cdot67\ Hz$$

$\omega = 2\pi f = 2\pi \times 9{\cdot}67 = 61$ rad s^{-1}

$$E_k = \frac{1}{2}m\omega^2(A^2 - y^2)$$

$$E_k = \frac{1}{2} \times 3\cdot67 \times 61^2 \times (0\cdot013^2 - 0)$$

$\underline{E_k = 1{\cdot}2\ J}$

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Worked example 3 — 2019 Q11(b)

A coin on the lid of the same tin loses contact when the tin's downward acceleration reaches $9{\cdot}8\ \mathrm{m\ s^{-2}}$. Find that displacement.

$a = 9{\cdot}8\ \mathrm{m\ s^{-2}}$

$\omega = 61\ \mathrm{rad\ s^{-1}}$

$y = ?$

$$a = -\omega^2 y$$

$$y = \frac{a}{\omega^2}$$

$$y = \frac{9 \cdot 8}{61^2}$$

$\underline{y = 2{\cdot}6 \times 10^{-3}\ \mathrm{m}}$

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PERIOD 4 · THE PERIOD OF AN OSCILLATOR

How long is one oscillation?

One method, and it works for every system.

JOTTER NOTES

Finding T for any SHM system

The same four steps work for every oscillator:

1. Show the restoring force is proportional to $-y$.
2. Apply $F = ma$ to get the acceleration.
3. Equate it to $a = -\omega^2 y$ and read off ω^2.
4. Use $T = 2\pi/\omega$ to get the period.

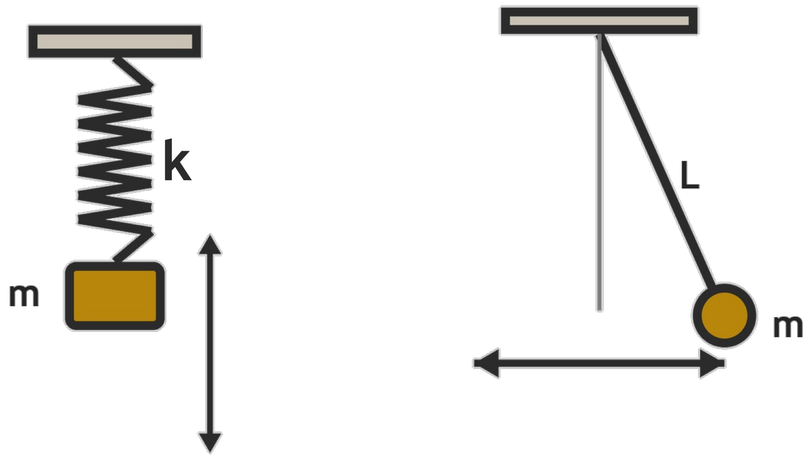
The simple pendulum

$y = L\theta$	y is measured along the arc.
$F = -mg\, sin\, \theta$	The weight's component along the arc.
$F = -mg\theta$	Small angle in radians: sin θ ≈ θ.
$a = -g\dfrac{y}{L}$	F = ma, then put θ = y/L.
$\omega^2 = \dfrac{g}{L}$	Compare with a = $-\omega^2$y.

T = 2π/ω gives T = 2π√(L/g), for small θ only.

Two special cases

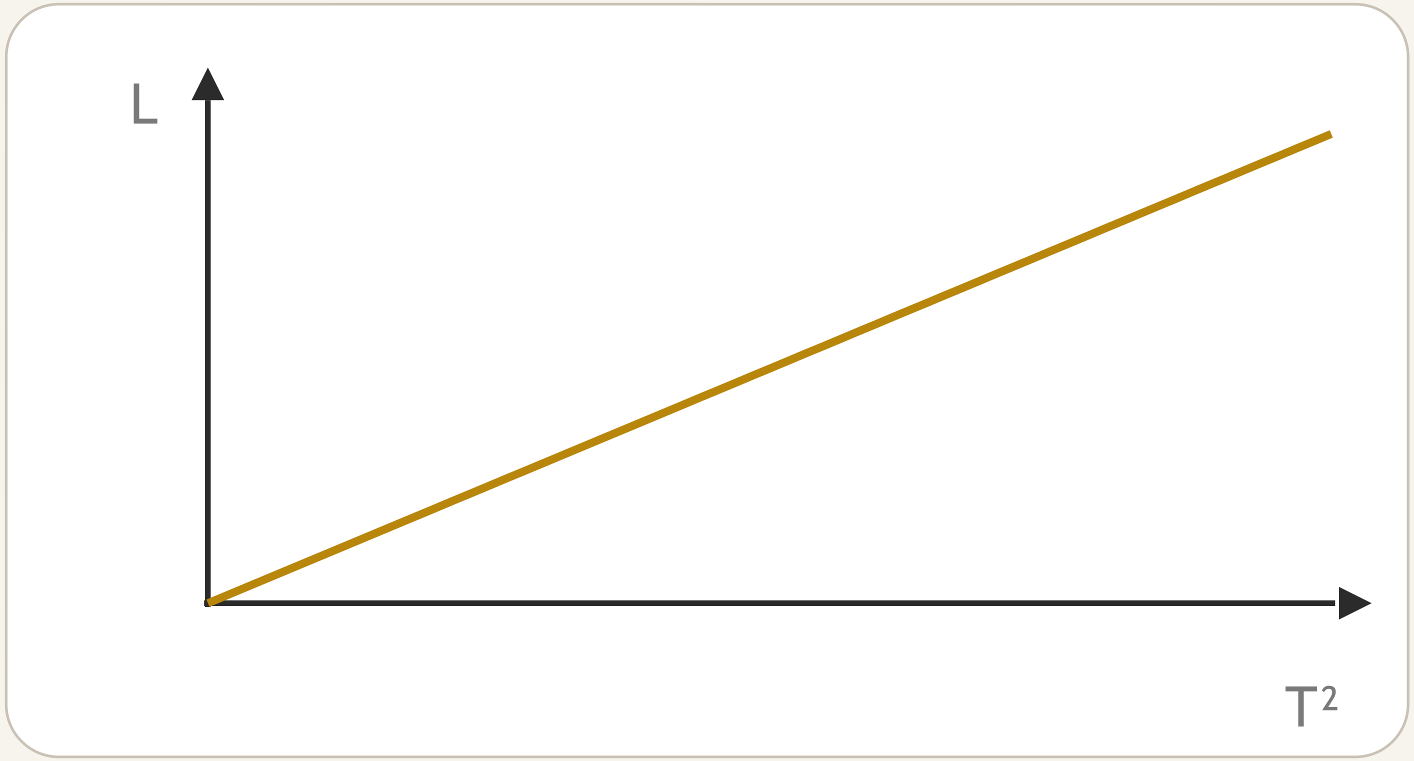
$$T = 2\pi\sqrt{\frac{m}{k}} \qquad T = 2\pi\sqrt{\frac{L}{g}}$$



Each equation belongs to the system drawn beneath it.

Finding g from a T^2 graph

L against T^2 is a straight line through the origin



Squaring $T = 2\pi\sqrt{(L/g)}$ gives $T^2 = 4\pi^2 L/g$, so the gradient of L against T^2 is $g/4\pi^2$ — and that is how the pendulum experiment measures g.

PERIOD 5 · DAMPING, AND REVIEW

What friction does to it

No real oscillator runs forever.

JOTTER NOTES

Damping

In a real system, friction transfers energy out of the system.

The total energy of an oscillator is $E_{tot} = \frac{1}{2}m\omega^2A^2$.

So as the energy falls, the amplitude of the motion falls too.

How quickly that happens is the damping of the system.

JOTTER NOTES

Underdamped, critically damped, overdamped

Underdamped: the system oscillates, and the amplitude dies away.

Critically damped: it does not oscillate at all, and it reaches rest in the shortest possible time.

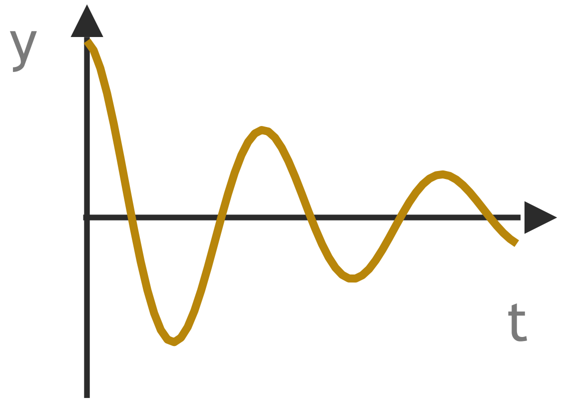
Overdamped: it does not oscillate either, but takes longer to come to rest than a critically damped system does.

Three kinds of damping

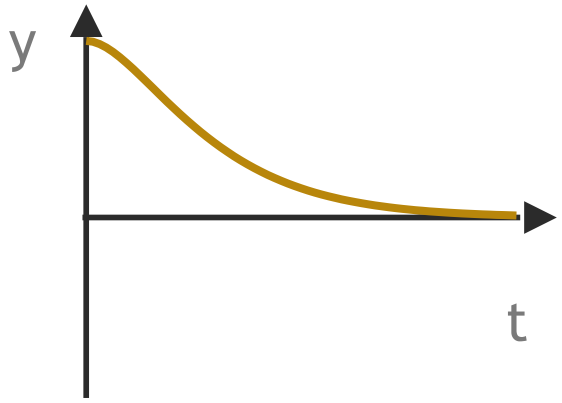
JOTTER NOTES

Critical damping reaches rest quickest, without oscillating

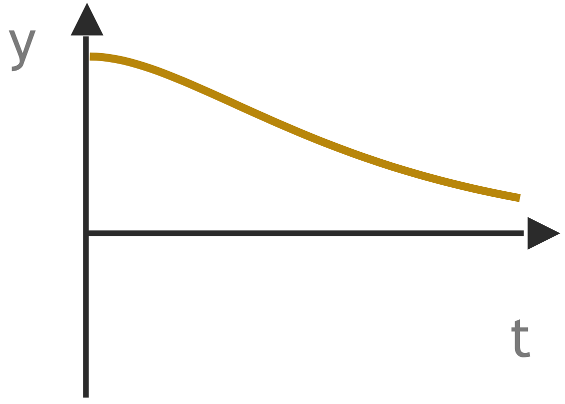
Underdamped



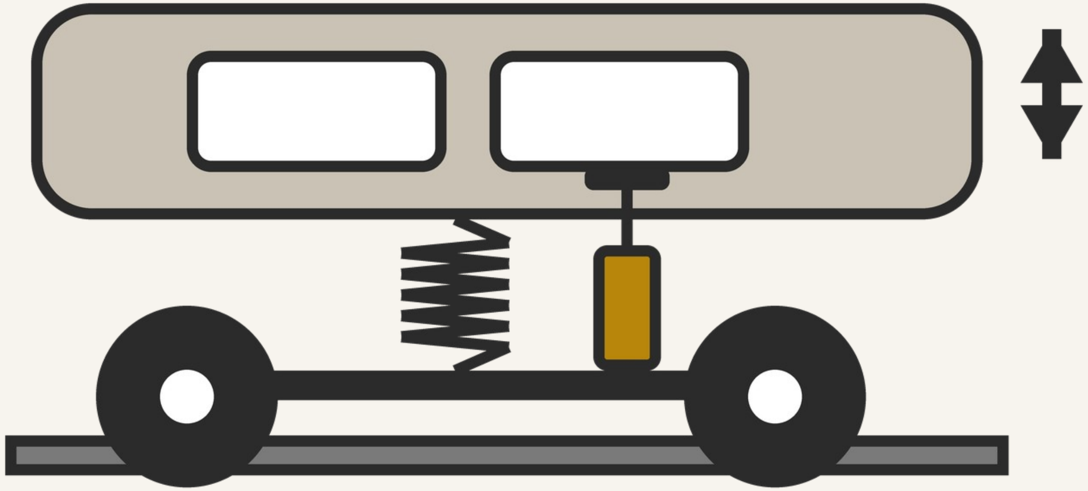
Critical



Overdamped



Why critical damping is the one you want



Worn shock absorbers underdamp, and the car keeps bouncing.

SHM in the exam

A full question in every diet since 2016 — usually Q8 to Q11

Diet	Question	What it asked
2016	Q10	Define SHM; the cosine solution
2017	Q8	Choosing the right model
2020	Q9	Define SHM; the sine solution
2023	Q10(a), (b), (f)	Calculation
2024	Q9	Calculation
2025	Q9(b), (c)	Calculation

2018, 2019 and 2022 are worked in class above.

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Reading and homework

Scholar: Quanta and waves, Topic 4 — Simple harmonic motion

Read the topic, then sit the Scholar end-of-topic test.

Scholar — Quanta and waves, Topic 4
Scholar — end-of-topic test (homework)
BrightRed Study Guide pp. 70-75
Past papers — 2016, 2017, 2020, 2023-25

Scholar

JOTTER NOTES

Errors that cost marks

The minus sign in $a = -\omega^2 y$ is part of the answer, not decoration.

a is not constant, so the suvat equations never apply to SHM.

The amplitude A is the maximum displacement, not the full swing.

ωt is in radians — check the calculator is not in degrees.

A definition with no mention of direction scores zero.

Check yourself

Can you do each of these without your notes?

1. State the definition of SHM, word for word.
2. Derive $a = -\omega^2 y$ from $y = A \sin \omega t$, showing every step.
3. Derive $v = \pm\omega\sqrt{(A^2 - y^2)}$ and $E_k = \frac{1}{2}m\omega^2(A^2 - y^2)$.
4. Derive $T = 2\pi\sqrt{(L/g)}$ for a simple pendulum.
5. Describe underdamping, critical damping and overdamping.

Experiments you could try

Standard kit, or your own phone

- Mass on a spring with a motion sensor: plot T^2 against mass.
- Simple pendulum: plot T^2 against length, and find g.
- Trolley between two springs: take the y–t graph off a sensor.
- Loaded test tube bobbing in water: time 20 oscillations.
- Damping: add a card to the mass, then repeat it in water.
- DIY: a phone as accelerometer, or film the bob at 240 fps.

Sources & credits

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